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Stabilization of Interval Type-2 Polynomial-Fuzzy-Model-Based Control Systems

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***Abstract*—In this paper, the stability of polynomial-fuzzy-model-based (PFMB) systems equipped with mismatched interval type-2 (IT2) membership functions is investigated. Unlike the membership-function-independent methods, the information and properties of IT2 membership functions are considered in the stability analysis and contained in the stability conditions in terms of sum-of-squares (SOS) based on the Lyapunov stability theory. Three methods, demonstrating their own advantages, are proposed to conduct the stability analysis for the IT2 PFMB control systems. In the first one, we divide the operating domain into subdomains and then conduct the stability analysis incorporating the information and properties of the IT2 membership functions in subdomains. Through this approach, the stability conditions can be further relaxed compared with the membership-function-independent analysis. Polynomial functions are adopted in the second method to approximate the IT2 membership functions. The advantage of this method compared with the first one is that richer information of IT2 membership functions is considered without increasing the number of SOS conditions. In the third one, we combine the advantages of both the first and the second method offering a new approach which utilizes the information and properties of the lower and upper IT2 membership functions in subdomains through simpler polynomial approximation functions. It can be shown that more relaxed stability conditions can be obtained compared with the first two methods. Numerical examples and simulations are presented to verify the effectiveness of the proposed methods.**

***Index Terms*—Interval type-2 (IT2) fuzzy logic, polynomial fuzzy-model-based (PFMB) control systems, stability analysis, sum of squares (SOS).**

I. Introduction

A. Background Knowledge

TYPE-1 fuzzy set theory was first proposed by Zadeh in 1965 [1], which has been widely applied in domestic and industrial fuzzy control approaches. One of the most important control approaches is the fuzzy-model-based (FMB) control approach. It is well known that Takagi–Sugeno (T–S) fuzzy model [2] plays an important role in FMB control systems for its capability to provide general modeling frameworks for nonlinear systems. Besides, thanks to its rigorous mathematical structure, there are systematic ways to carry out stability analysis and control synthesis [2]–[6], which are considered as the most important issues to be addressed in the FMB control systems.

Stability analysis is one of the most important parts of the control design, and the Lyapunov stability theory is one of the most popular methods to investigate the stability of T–S FMB control systems. According to the Lyapunov approach, if there exists a common solution to all Lyapunov inequalities in terms of linear matrix inequalities (LMIs), the T–S FMB control system is guaranteed to be asymptotically stable [7]. Considering the feedback control, the most popular design method is parallel distributed compensation (PDC) [7], which was developed based on the idea that both the plant and controller share the same premise fuzzy rule set. There are a lot of works managed to further relax the PDC approach-based stability conditions [2]–[5], [8], [9] and generalized by applying Pólya's theorem [10].

Given that the controller is required to share the same rule set with the plant in the PDC approach, in general, the design flexibility is reduced, and the implementation cost is also increased. In order to render the system flexibly and lower the implementation cost, it makes sense to consider the case that the fuzzy model and fuzzy controller do not share the same premise fuzzy rule set, which results in imperfectly matched membership functions [11], [12]. It should be noted that when the requirement of the same rule set is removed, the results of stability analysis can be very conservative as the permutations of membership functions used in the PDC design approach cannot be applied due to the imperfectly matched membership functions. In addition, in most of the related works, the shapes of membership functions have not been considered during the analysis, which means that the stability conditions are valid unnecessarily for any arbitrary membership functions and, hence, result in conservativeness. As the stability conditions only need to be valid under the specific membership functions used in the investigated fuzzy plant and fuzzy controller, bringing the information of membership functions into the analysis contributes to the relaxation of stability conditions. In [11]–[13], the local/global boundary information of membership functions was employed to relax the stability conditions. In [14], staircase-shaped functions were adopted to approximate the original membership functions in the stability analysis of FMB control systems, which allows adding the approximated membership functions into the stability conditions to make them membership function dependent, which leads to more relaxed stability analysis results. Along this line,

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piecewise-linear membership functions [15] and Taylor-series membership functions [16] were proposed to carry more information to facilitate the stability analysis.

As an extension of the T–S fuzzy model, the polynomial fuzzy model has been proposed recently [17]. Instead of only considering linear terms in the consequent part in the T–S fuzzy model, the polynomial fuzzy model adopts polynomial terms. When all the polynomial terms are zeroth-order polynomials, the polynomial fuzzy model is reduced to the T–S fuzzy model. Therefore, the polynomial fuzzy model has more potential to precisely represent nonlinear systems over the T–S fuzzy model. However, due to the introduction of polynomial terms, the LMI approach used for the T–S FMB fuzzy control can no longer be used to conduct the stability analysis. Instead, the sum-of-squares (SOS) approach is widely used in the stability analysis of polynomial-fuzzy-model-based (PFMB) control systems. Based on polynomial Lyapunov functions that contain quadratic Lyapunov functions as a special case, the stability conditions are derived in form of SOS, which can be solved efficiently through a third-party MATLAB toolbox SOSTOOLS [18]. Compared with the T–S FMB control systems, results for SOS-based stability analysis are relatively much less, but can also be found such as in [13], [15], [17], and [19]–[22].

Type-1 fuzzy set is able to deal with the nonlinearities in control systems, but lacks the capability to handle the uncertainties directly, since the membership functions do not contain any uncertain information [23], [24]. Quite often, there are lots of inevitable uncertainties, which can be found during the construction of the rules in FMB control systems. In general, the uncertainties can be classified into two types, namely, the linguistic uncertainties and random uncertainties [23]. In order to include the uncertainties into the type-1 membership functions, the concept of footprint of uncertainty (FOU) has been introduced into the type-1 membership functions, which render type-1 fuzzy systems into type-2 fuzzy systems [23].

In terms of type-2 membership functions, there are huge complexities embedded in the FOU, which results in difficult stability analysis and high computational burden on the numerical simulations. Therefore, the widely used type-2 fuzzy systems are based on interval type-2 (IT2) membership functions instead of the general type-2 membership functions. It is worth mentioning that the type-2 fuzzy sets can also be considered as the generalization of interval-valued fuzzy sets [25], and interval-valued fuzzy sets are a particular case of the IT2 fuzzy sets [26]. Regarding the IT2 membership functions, all membership grades of the secondary membership functions are constants instead of functions of premise variables. Although the compromise on complexity and performance has been made, by adopting IT2 membership functions, we can not only handle the uncertainties directly, but reduce the computational burden as well [23], [27]–[29]. Recently, the research has been conducted on the system control and stability analysis based on the framework of IT2 fuzzy systems [30]–[34]. To the best knowledge of the authors' knowledge, although there has been some research on the IT2 fuzzy control systems, the issues of stability analysis and control synthesis on IT2 PFMB control systems are rarely investigated.

B. Research Methodology

In this paper, the stability conditions of the IT2 PFMB control system are obtained through the SOS approach, and the imperfectly matched IT2 membership functions are also considered for the purpose of giving more flexibility to the IT2 fuzzy controller. It is worth mentioning that the IT2 controllers used to stabilize the control systems are designed systematically through the SOS-based approaches, and the information of membership functions is also considered during the stability analysis. Given that the IT2 membership functions are continuous in general cases, when taken into the stability analysis, it will lead to infinite stability conditions that are not practical to be solved numerically. To address this difficulty, we divide the whole domain of the IT2 membership functions into some subdomains. This way, there are only finite situations need to be considered in the stability analysis and then make it possible to solve numerically the SOS-based stability conditions. In addition, polynomial-function-approximation methods are introduced to ease the SOS-based stability analysis. To further relax the stability conditions, slack matrices are used to include more information of the IT2 membership functions into the stability conditions. In addition, we combine the advantages of both methods to develop the third one, which can relax the stability conditions to the most from the numerical simulation results.

The rest of this paper is organized as follows. Section II briefly presents the preliminaries of IT2 polynomial fuzzy models and controllers. In Section III, the stability issues of the IT2 PFMB control system have been discussed. Three methods are established, and the related stability analysis is investigated based on the Lyapunov stability theory. In Section IV, simulation examples are given to illustrate the advantages of the proposed stability analysis methods. Conclusions are drawn in Section V.

II. Interval Type-2 Polynomial Fuzzy Model and Fuzzy Controller

A. Interval Type-2 polynomial Fuzzy Model

An IT2 polynomial fuzzy model with p rules, extended from [24] and [35], is employed to describe the dynamics of the nonlinear plant. The rules are of the following format where the antecedents are IT2 fuzzy sets, and the consequent is a linear dynamic system:

$$\begin{aligned}\text{Rule } i:\ &\text{IF } f_1(\mathbf{x}(t)) \text{ is } \tilde{M}_1^i \text{ AND } \cdots \text{ AND } f_\Psi(\mathbf{x}(t)) \text{ is } \tilde{M}_\Psi^i \\ &\text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t)\end{aligned} \tag{1}$$

where $\tilde{M}_\alpha^i$ is a fuzzy term of rule i corresponding to the known function $f_\alpha(\mathbf{x}(t))$, $\alpha = 1, 2, ..., \Psi$ and $i = 1, 2, \ldots, p$; Ψ is a positive integer; $\mathbf{A}_i(\mathbf{x}(t)) \in \Re^{n\times N}$ and $\mathbf{B}_i(\mathbf{x}(t)) \in \Re^{n\times m}$ are known polynomial system and input matrices; $\mathbf{x}(t) \in \Re^n$ is the system-state vector; $\hat{\mathbf{x}}(\mathbf{x}(t)) \in \Re^N$ is a vector of monomials in $\mathbf{x}(t)$; and $\mathbf{u}(t) \in \Re^m$ is the control input vector. The firing strength of the ith rule is within the following interval sets:

$$\tilde{w}_i(\mathbf{x}(t)) \in [w_i^L(\mathbf{x}(t)), w_i^U(\mathbf{x}(t))], \quad i = 1, 2, \ldots, p \tag{2}$$

where $w_i^L(\mathbf{x}(t)) = \prod_{l=1}^{\Psi} \underline{\mu}_{\tilde{M}_l^i}(f_l(\mathbf{x}(t))), w_i^U(\mathbf{x}(t)) = \prod_{l=1}^{\Psi} \overline{\mu}_{\tilde{M}_l^i}(f_l(\mathbf{x}(t)))$, in which $0 \le \overline{\mu}_{\tilde{M}_\alpha^i}(f_\alpha(\mathbf{x}(t))) \le 1$ and $0 \le \underline{\mu}_{\tilde{M}_\alpha^i}(f_\alpha(\mathbf{x}(t))) \le 1$ denote the lower and upper grades of membership governed by their lower and upper membership functions, respectively. By the definition of IT2 membership functions, the property $0 \le \underline{\mu}_{\tilde{M}_\alpha^i}(f_\alpha(\mathbf{x}(t))) \le \overline{\mu}_{\tilde{M}_\alpha^i}(f_\alpha(\mathbf{x}(t))) \le 1$ holds, which further leads to $0 \le w_i^L(\mathbf{x}(t)) \le w_i^U(\mathbf{x}(t)) \le 1$ for all i.

We define $\tilde{w}_i(\mathbf{x}(t))$ as $\tilde{w}_i(\mathbf{x}(t)) = \underline{\lambda}_i(\mathbf{x}(t)) w_i^L(\mathbf{x}(t)) + \overline{\lambda}_i(\mathbf{x}(t)) w_i^U(\mathbf{x}(t)$, in which $0 \le \underline{\lambda}_i(\mathbf{x}(t)) \le 1, 0 \le \overline{\lambda}_i(\mathbf{x}(t)) \le 1, \underline{\lambda}_i(\mathbf{x}(t)) + \overline{\lambda}_i(\mathbf{x}(t)) = 1, \forall\, i$. $\underline{\lambda}_i(\mathbf{x}(t))$ and $\overline{\lambda}_i(\mathbf{x}(t))$ are nonlinear functions to be determined.

The IT2 polynomial fuzzy model is described by

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} \tilde{w}_i(\mathbf{x}(t))(\mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(t) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t)) \tag{3}$$

where

$$\sum_{i=1}^{p} \tilde{w}_i(\mathbf{x}(t)) = 1, \tilde{w}_i(\mathbf{x}(t)) \ge 0 \quad \forall\, i. \tag{4}$$

B. Interval Type-2 Polynomial-Fuzzy-Model-Based Controller

An IT2 polynomial fuzzy controller with c rules is employed to stabilize the plant represented by the IT2 polynomial fuzzy model (3). The format of the IT2 polynomial fuzzy controller is as follows:

$$\begin{aligned} \text{Rule } j: \ & \text{IF } g_1(\mathbf{x}(t)) \text{ is } \tilde{N}_1^j \text{ AND } \cdots \text{ AND } g_\Omega(\mathbf{x}(t)) \text{ is } \tilde{N}_\Omega^j \\ & \text{THEN } \mathbf{u}(t) = \mathbf{G}_j(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) \end{aligned} \tag{5}$$

where $\tilde{N}_\beta^j$ is an IT2 fuzzy term of rule j corresponding to function $g_\beta(\mathbf{x}(t))$, where $\beta = 1, 2, \ldots, \Omega$ and $j = 1, 2, \ldots, c$, Ω is a positive integer, and $\mathbf{G}_j(\mathbf{x}(t)) \in \Re^{m\times N}$, $j = 1, 2, \ldots, c$, is the polynomial feedback gain to be determined. The firing strength of the jth rule is within the following interval sets:

$$\tilde{m}_j(\mathbf{x}(t)) \in [m_j^L(\mathbf{x}(t)), m_j^U(\mathbf{x}(t))], \quad j = 1, 2, \ldots, c \tag{6}$$

where $m_j^L(\mathbf{x}(t)) = \prod_{l=1}^{\Omega} \underline{\mu}_{\tilde{N}_l^j}(g_l(\mathbf{x}(t))), m_j^U(\mathbf{x}(t)) = \prod_{l=1}^{\Omega} \overline{\mu}_{\tilde{N}_l^j}(g_l(\mathbf{x}(t)))$, in which $0 \le \overline{\mu}_{\tilde{N}_\beta^j}(g_\beta(\mathbf{x}(t))) \le 1$ and $0 \le \underline{\mu}_{\tilde{N}_\beta^i}(g_\beta(\mathbf{x}(t))) \le 1$ denote the lower and upper grades of membership governed by the lower and upper membership functions, respectively. By the definition of IT2 membership functions, the property $0 \le \underline{\mu}_{\tilde{N}_\beta^j}(g_\beta(\mathbf{x}(t))) \le \overline{\mu}_{\tilde{N}_\beta^j}(g_\beta(\mathbf{x}(t))) \le 1$ holds and further leads to the $0 \le m_j^L(\mathbf{x}(t)) \le m_j^U(\mathbf{x}(t)) \le 1$ valid for all j.

In addition, we define $\tilde{m}_j(\mathbf{x}(t))$ as follows: $\tilde{m}_j(\mathbf{x}(t)) = \underline{\kappa}_j(\mathbf{x}(t)) m_j^L(\mathbf{x}(t)) + \overline{\kappa}_j(\mathbf{x}(t)) m_j^U(\mathbf{x}(t), 0 \le \underline{\kappa}_j(\mathbf{x}(t)) \le 1, 0 \le \overline{\kappa}_j(\mathbf{x}(t)) \le 1, \underline{\kappa}_j(\mathbf{x}(t)) + \overline{\kappa}_j(\mathbf{x}(t)) = 1, \ \forall\, j$. $\underline{\kappa}_j(\mathbf{x}(t))$ and $\overline{\kappa}_j(\mathbf{x}(t))$ are nonlinear functions to be determined.

The IT2 polynomial fuzzy controller is described by

$$\mathbf{u}(t) = \sum_{i=1}^{c} \tilde{m}_j(\mathbf{x}(t))\mathbf{G}_j(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) \tag{7}$$

where

$$\sum_{i=1}^{c} \tilde{m}_j(\mathbf{x}(t)) = 1, \tilde{m}_j(\mathbf{x}(t)) \ge 0 \quad \forall\, j. \tag{8}$$

III. Stability Analysis of Interval Type-2 Polynomial-Fuzzy-Model-Based Systems

The stability analysis of the IT2 PFMB systems is investigated in this section. In the following analysis, for brevity, the time t associated with the variables is dropped for the situation without ambiguity, e.g., $\mathbf{x}(t)$ and $\hat{\mathbf{x}}(\mathbf{x}(t))$ are denoted as $\mathbf{x}$ and $\hat{\mathbf{x}}$, respectively. In addition, $\tilde{w}_i(\mathbf{x}(t))$ and $\tilde{m}_j(\mathbf{x}(t))$ are denoted as $\tilde{w}_i$ and $\tilde{m}_j$, respectively. From (1) and (5), we obtain the IT2 PFMB control system as follows:

$$\dot{\mathbf{x}} = \sum_{i=1}^{p}\sum_{j=1}^{c} \tilde{w}_i \tilde{m}_j (\mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{G}_j(\mathbf{x}))\hat{\mathbf{x}}. \tag{9}$$

From (9), denoting $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$ and $\hat{\mathbf{x}} = [\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_N]^T$, we have

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= \frac{\partial \hat{\mathbf{x}}}{\partial \mathbf{x}} \frac{d\mathbf{x}}{dt} = \mathbf{T}(\mathbf{x})\dot{\mathbf{x}} \\ &= \sum_{i=1}^{p}\sum_{j=1}^{c} \tilde{w}_i \tilde{m}_j (\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{G}_j(\mathbf{x}))\hat{\mathbf{x}} \end{aligned} \tag{10}$$

where $\tilde{\mathbf{A}}_i(\mathbf{x}) = \mathbf{T}(\mathbf{x})\mathbf{A}_i(\mathbf{x})$, $\tilde{\mathbf{B}}_i(\mathbf{x}) = \mathbf{T}(\mathbf{x})\mathbf{B}_i(\mathbf{x})$, and $\mathbf{T}(\mathbf{x}) \in \Re^{N\times N}$ is a polynomial matrix with (i, j)th element is defined as $T_{i,j} = \partial \hat{x}_i(\mathbf{x})/\partial x_j$. Since $\hat{\mathbf{x}}$ is a vector of monomials of $\mathbf{x}$, $\hat{\mathbf{x}} = 0$ implies $\mathbf{x} = 0$; therefore, the stability of the augmented IT2 PFMB control system (10) implies that of the IT2 PFMB control system (9).

A. Sum-of-Squares-Based Stability Analysis

The following polynomial Lyapunov function candidate is employed to investigate the stability of the augmented IT2 PFMB control system (10):

$$V(t) = \hat{\mathbf{x}}^T \mathbf{X}(\tilde{\mathbf{x}})^{-1} \hat{\mathbf{x}} \tag{11}$$

where $0 < \mathbf{X}(\tilde{\mathbf{x}}) = \mathbf{X}(\tilde{\mathbf{x}})^T \in \Re^{N\times N}$.

Remark 1: To facilitate the stability analysis, it is defined that $\mathbf{K} = \{k_1, k_2, \ldots, k_q\}$ is the set of row numbers that the entire row of $\mathbf{B}_i(\mathbf{x})$ are all zeros for all i. Defining $\tilde{\mathbf{x}} = (x_{k_1}, x_{k_2}, \ldots, x_{k_q})$, it obtains $\frac{\partial \mathbf{X}(\tilde{\mathbf{x}})^{-1}}{\partial x_j} = -\mathbf{X}(\tilde{\mathbf{x}})^{-1} \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial x_j} \mathbf{X}(\tilde{\mathbf{x}})^{-1}$ and $\dot{\mathbf{X}}(\tilde{\mathbf{x}})^{-1} = \sum_{k\in\mathbf{K}} \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})^{-1}}{\partial x_k} \sum_{i=1}^{p} \tilde{w}_i \mathbf{A}_i^k(\mathbf{x})\hat{\mathbf{x}}$ [17], where $\mathbf{A}_i^k(\mathbf{x}) \in \Re^N$ and $\mathbf{B}_i^k(\mathbf{x}) \in \Re^m$, $i = 1, 2, \ldots, p$, $k = 1, 2, \ldots, n$ denote the kth row of $\mathbf{A}_i(\mathbf{x})$ and $\mathbf{B}_i(\mathbf{x})$, respectively.

From (10) and (11), we have

$$\begin{aligned}\dot{V}(t) &= \dot{\hat{\mathbf{x}}}^T \mathbf{X}(\tilde{\mathbf{x}})^{-1}\hat{\mathbf{x}} + \hat{\mathbf{x}}^T \mathbf{X}(\tilde{\mathbf{x}})^{-1}\dot{\hat{\mathbf{x}}} + \hat{\mathbf{x}}^T \frac{d\mathbf{X}(\tilde{\mathbf{x}})^{-1}}{\mathrm{dt}}\hat{\mathbf{x}} \\ &= \sum_{i=1}^{p}\sum_{j=1}^{c} \tilde{w}_i \tilde{m}_j \hat{\mathbf{x}}^T \Big(\big(\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{G}_j(\mathbf{x})\big)^T \mathbf{X}(\tilde{\mathbf{x}})^{-1} \\ &\quad + \mathbf{X}(\tilde{\mathbf{x}})^{-1}\big(\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{G}_j(\mathbf{x})\big)\Big)\hat{\mathbf{x}} \\ &\quad + \hat{\mathbf{x}}^T \frac{d\mathbf{X}(\tilde{\mathbf{x}})^{-1}}{\mathrm{dt}}\hat{\mathbf{x}}. \end{aligned} \tag{12}$$

Let us denote $\tilde{w}_i \tilde{m}_j$ as $\tilde{h}_{ij}(\mathbf{x})$ and define $\mathbf{z} = \mathbf{X}(\tilde{\mathbf{x}})^{-1}\hat{\mathbf{x}}$ and $\mathbf{G}_j(\mathbf{x}) = \mathbf{N}_j(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}})^{-1}$, where $\mathbf{N}_j(\mathbf{x}) \in \Re^{m\times N}$, $j = 1, 2, \ldots, c$, is an arbitrary polynomial matrix to be determined. From Remark 1 and (10)–(12), we have

$$\dot{V}(t) = \sum_{i=1}^{p}\sum_{j=1}^{c} \tilde{h}_{ij}(\mathbf{x})\mathbf{z}^T \mathbf{Q}_{ij}(\mathbf{x})\mathbf{z} \tag{13}$$

where $\mathbf{Q}_{ij}(\mathbf{x}) = \tilde{\mathbf{A}}_i\mathbf{X}(\tilde{\mathbf{x}}) + \mathbf{X}(\tilde{\mathbf{x}})\tilde{\mathbf{A}}_i(\mathbf{x})^T + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{N}_j(\mathbf{x}) + \mathbf{N}_j(\mathbf{x})^T\tilde{\mathbf{B}}_i(\mathbf{x})^T - \sum_{k\in\mathbf{K}} \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial x_k}\mathbf{A}_i^k(\mathbf{x})\hat{\mathbf{x}}$, $i = 1, 2, \ldots, p$; $j = 1, 2, \ldots, c$.

Remark 2: For IT2 PFMB control systems, the membership grades $\tilde{w}_i$ for all i are uncertain, which hinder the stability analysis using the techniques requiring the membership functions to be known, e.g., the PDC design. The most straightforward approach to guarantee the stability of the control systems is to require $\mathbf{X}(\tilde{\mathbf{x}}) > 0$ and $\mathbf{Q}_{ij}(\mathbf{x}) < 0$ for all i and j. According to Lyapunov stability theory, by satisfying these conditions, $V(t) > 0$ and $\dot{V}(t) < 0$ (excluding $\mathbf{x} = \mathbf{0}$) can be achieved, which implies the asymptotic stability of (9). However, the stability conditions will be very conservative, as the membership functions $\tilde{h}_{ij}(\mathbf{x})$ are not considered in the stability analysis, which means that the stability conditions are unnecessarily valid for arbitrary membership functions. In order to include the specific membership functions into the analysis, some basic techniques were proposed in [24] and [35] to utilize limited information of membership functions.

B. Subdomains of Membership Functions

It is worth noting that $\tilde{h}_{ij}(\mathbf{x})$ is a function of $\mathbf{x}$, which has an infinite number of membership grades due to the continuous variable $\mathbf{x}$. Consequently, by incorporating the membership functions into the stability conditions, it is not practical to find a feasible solution to the stability conditions of infinite number. In this paper, we propose various techniques to bring the information of membership functions into the stability analysis, which avoids turning the number of stability conditions into infinite, but still can achieve more relaxed stability conditions.

To facilitate the stability analysis, we first divide the whole operating domain Φ into L connected subdomains, Φ_l, $l = 1, 2, \ldots, L$, such that $\Phi = \bigcup_{l=1}^{L}\Phi_l$. In each subdomain, we denote the portion of $\tilde{h}_{ij}(\mathbf{x})$, where $\mathbf{x} \in \Phi_l$ (the portion of $\tilde{h}_{ij}(\mathbf{x})$ in the lth subdomain) as $\tilde{h}_{ijl}(\mathbf{x})$ such that $\tilde{h}_{ij}(\mathbf{x}) = \bigcup_{l=1}^{L}\tilde{h}_{ijl}(\mathbf{x})$.

In the following, we conduct the stability analysis subdomain by subdomain by utilizing the information of $\tilde{h}_{ijl}(\mathbf{x})$ for $\mathbf{x} \in \Phi_l$.

Equation (13) is rewritten in the lth subdomain as follows:

$$\begin{aligned}\dot{V}(t) &= \sum_{i=1}^{p}\sum_{j=1}^{c} \tilde{h}_{ijl}(\mathbf{x})\mathbf{z}^T\mathbf{Q}_{ij}(\mathbf{x})\mathbf{z} \\ &= \sum_{i=1}^{p}\sum_{j=1}^{c} (\hat{h}_{ijl} + \tilde{h}_{ijl}(\mathbf{x}) - \hat{h}_{ijl})\mathbf{z}^T\mathbf{Q}_{ij}(\mathbf{x})\mathbf{z}, \mathbf{x} \in \Phi_l \\ l &= 1, 2, \ldots, L \end{aligned} \tag{14}$$

where $\hat{h}_{ijl} \geq 0$ is a constant, which is an estimate of $h_{ijl}(\mathbf{x})$ to be determined. Meanwhile, we define some nonnegative matrices $\mathbf{Y}_{ijl}(\mathbf{x}) = \mathbf{Y}_{ijl}(\mathbf{x})^T \geq 0$, which are required to satisfy $\mathbf{Y}_{ijl}(\mathbf{x}) \geq \mathbf{Q}_{ij}(\mathbf{x})$. From (14), we have

$$\begin{aligned}\dot{V}(t) &= \sum_{i=1}^{p}\sum_{j=1}^{c} \hat{h}_{ijl}\mathbf{z}^T\mathbf{Q}_{ij}(\mathbf{x})\mathbf{z} + \sum_{i=1}^{p}\sum_{j=1}^{c}(\tilde{h}_{ijl} - \hat{h}_{ijl})\mathbf{z}^T\mathbf{Q}_{ij}(\mathbf{x})\mathbf{z} \\ &\leq \sum_{i=1}^{p}\sum_{j=1}^{c} \hat{h}_{ijl}\mathbf{z}^T\mathbf{Q}_{ij}(\mathbf{x})\mathbf{z} \\ &\quad + \sum_{i=1}^{p}\sum_{j=1}^{c} |\tilde{h}_{ijl}(\mathbf{x}) - \hat{h}_{ijl}|\mathbf{z}^T\mathbf{Y}_{ijl}(\mathbf{x})\mathbf{z} \\ &= \sum_{i=1}^{p}\sum_{j=1}^{c} \mathbf{z}^T(\hat{h}_{ijl}\mathbf{Q}_{ij}(\mathbf{x}) + |\tilde{h}_{ijl}(\mathbf{x}) - \hat{h}_{ijl}|\mathbf{Y}_{ijl}(\mathbf{x}))\mathbf{z}, \\ &\quad \mathbf{x} \in \Phi_l, l = 1, 2, \ldots, L. \end{aligned} \tag{15}$$

To proceed further, we define constant scalars $\underline{h}_{ijl} \geq 0$ and $\overline{h}_{ijl} \geq 0$ as the lower and upper bounds of the IT2 membership function $\tilde{h}_{ijl}(\mathbf{x})$ in the lth subdomains, respectively, satisfying $0 \leq \underline{h}_{ijl} \leq \tilde{h}_{ijl}(\mathbf{x}) \leq \overline{h}_{ijl} \leq 1$ for $\mathbf{x} \in \Phi_l$. Choosing the constant scalar $\hat{h}_{ijl}$ satisfying $0 \leq \underline{h}_{ijl} \leq \hat{h}_{ijl} \leq \overline{h}_{ijl} \leq 1$ for $\mathbf{x} \in \Phi_l$, we obtain $\overline{h}_{ijl} - \underline{h}_{ijl} \geq |\tilde{h}_{ijl}(\mathbf{x}) - \hat{h}_{ijl}| \geq 0$. Therefore, through adopting $\underline{h}_{ijl}$, $\overline{h}_{ijl}$, and $\hat{h}_{ijl}$ in the stability analysis, it is able to get rid of $\tilde{h}_{ijl}(\mathbf{x})$ to allow the stability conditions to be handled by convex programming techniques.

Since $\hat{h}_{ijl}$ can be any value in $[\underline{h}_{ijl}, \overline{h}_{ijl}]$, it is valid that $\max\{\tilde{h}_{ijl}(\mathbf{x}) - \underline{h}_{ijl}, \overline{h}_{ijl} - \tilde{h}_{ijl}(\mathbf{x})\} > |\tilde{h}_{ijl}(\mathbf{x}) - \hat{h}_{ijl}|$ for all subdomains. Then, (15) can be rewritten as follows:

$$V(t) \leq \sum_{i=1}^{p}\sum_{j=1}^{c} \mathbf{z}^T(\hat{h}_{ijl}\mathbf{Q}_{ij}(\mathbf{x}) + \delta_{ijl}\mathbf{Y}_{ijl}(\mathbf{x}))\mathbf{z} \tag{16}$$

where $\delta_{ijl} = \max\{\tilde{h}_{ijl}(\mathbf{x}) - \underline{h}_{ijl}, \overline{h}_{ijl} - \tilde{h}_{ijl}(\mathbf{x})\}$.

To further relax the stability analysis results, we bring the state information from each subdomain into the stability analysis. Defining the slack matrices $\mathbf{M}_l(\mathbf{x}) = \mathbf{M}_l^T(\mathbf{x}) \in \Re^{N\times N} \geq 0$, $l = 1, 2, \ldots, L$, it follows from (16) that

$$\begin{aligned}\dot{V}(t) \leq \sum_{i=1}^{p}\sum_{j=1}^{c} \mathbf{z}^T\big(&\hat{h}_{ijl}\mathbf{Q}_{ij}(\mathbf{x}) + \delta_{ijl}\mathbf{Y}_{ijl}(\mathbf{x}) \\ &+ (\mathbf{x} - \underline{\mathbf{x}}_l)^T\mathbf{D}(\overline{\mathbf{x}}_l - \mathbf{x})\mathbf{M}_l(\mathbf{x})\big)\mathbf{z} \end{aligned} \tag{17}$$

where $\underline{\mathbf{x}}_l \in \Re^N$ and $\overline{\mathbf{x}}_l \in \Re^N$ are the lower and upper bounds of $\mathbf{x}$ in the lth subdomain, $l = 1, 2, \ldots, L$; $\mathbf{D} =$

$\text{diag}\{d_1, d_2, \ldots, d_N\} \in \Re^{N\times N}$ is a diagonal matrix whose element is either 0 or 1. When $d_r = 0$, $r = 1, 2, \ldots, N$, the state information of x_r is not contained. Through the analysis, the results can be summarized as in the following theorem.

Theorem 1: The IT2 PFMB system (9), which is formed by a nonlinear plant represented by the IT2 polynomial fuzzy model (3) and the IT2 polynomial fuzzy controller (7) connected in a closed loop, is guaranteed to be asymptotically stable if there exist polynomial matrices $\mathbf{M}_l(\mathbf{x}) = \mathbf{M}_l(\mathbf{x})^T \in \Re^{N\times N}$, $\mathbf{N}_j(\mathbf{x}) \in \Re^{m\times N}$, $\mathbf{X}(\tilde{\mathbf{x}}) = \mathbf{X}(\tilde{\mathbf{x}})^T \in \Re^{N\times N}$, $\mathbf{Y}_{ijl}(\mathbf{x}) = \mathbf{Y}_{ijl}(\mathbf{x})^T \in \Re^{N\times N}$, $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, c$, $l = 1, 2, \ldots, L$, such that the following SOS-based conditions are satisfied:

$$
\begin{aligned}
&\nu^T(\mathbf{M}_l(\mathbf{x}) - \varepsilon_1(\mathbf{x})\mathbf{I})\nu \text{ is SOS} \quad \forall l\\
&\nu^T(\mathbf{X}(\tilde{\mathbf{x}}) - \varepsilon_2(\tilde{\mathbf{x}})\mathbf{I})\nu \text{ is SOS}\\
&\nu^T(\mathbf{Y}_{ijl}(\mathbf{x}) - \varepsilon_3(\mathbf{x})\mathbf{I})\nu \text{ is SOS} \quad \forall i, j, l\\
&\nu^T(\mathbf{Y}_{ijl}(\mathbf{x}) - \mathbf{Q}_{ij}(\mathbf{x}) - \varepsilon_4(\mathbf{x})\mathbf{I})\nu \text{ is SOS} \quad \forall i, j, l\\
&-\nu^T\sum_{i=1}^{p}\sum_{j=1}^{c}\big(\hat{h}_{ijl}\mathbf{Q}_{ij}(\mathbf{x}) + \delta_{ijl}\mathbf{Y}_{ijl}(\mathbf{x})\\
&+(\mathbf{x} - \underline{\mathbf{x}}_l)^T\mathbf{D}(\overline{\mathbf{x}}_l - \mathbf{x})\mathbf{M}_l(\mathbf{x}) + \varepsilon_5(\mathbf{x})\mathbf{I}\big)\nu \text{ is SOS} \quad \forall l
\end{aligned}
$$

where $\nu \in \Re^N$ is an arbitrary vector independent of $\mathbf{x}$; $\hat{h}_{ijl}$ and δ_{ijl} are predefined constants; $\mathbf{D} = \text{diag}\{d_1, d_2, \ldots, d_N\} \in \Re^{N\times N}$ is a predefined diagonal matrix; $\varepsilon_1(\mathbf{x}) > 0$, $\varepsilon_2(\tilde{\mathbf{x}}) > 0$, $\varepsilon_3(\mathbf{x}) > 0$, $\varepsilon_4(\mathbf{x}) > 0$, and $\varepsilon_5(\mathbf{x}) > 0$ are predefined scalar polynomials; $\underline{\mathbf{x}}_l$ and $\overline{\mathbf{x}}_l$ are the predefined lower and upper bounds of system state $\mathbf{x}$ in the lth subdomain; $\mathbf{Q}_{ij}(\mathbf{x}) = \tilde{\mathbf{A}}_i\mathbf{X}(\tilde{\mathbf{x}}) + \mathbf{X}(\tilde{\mathbf{x}})\tilde{\mathbf{A}}_i(\mathbf{x})^T + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{N}_j(\mathbf{x}) + \mathbf{N}_j(\mathbf{x})^T\tilde{\mathbf{B}}_i(\mathbf{x})^T - \sum_{k\in\mathbf{K}}\frac{\partial\mathbf{X}(\tilde{\mathbf{x}})}{\partial x_k}\mathbf{A}_i^k(\mathbf{x})\hat{\mathbf{x}}$; and the feedback gains are defined as $\mathbf{G}_j(\mathbf{x}) = \mathbf{N}_j(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}})^{-1}$, $j = 1, 2, \ldots, c$.

Remark 3: Referring to Theorem 1, the number of SOS variables is $pcL + L + c + 1$, and the number of SOS-based stability conditions is $2pcL + 2L + 1$. The more subdomains are divided, the richer information can be contained in the stability analysis, and then, the more relaxed stability analysis results can be achieved. However, when the number of subdomains increases, the number of stability conditions will also increase; therefore, the computational burden on solving the stability conditions will increase as well.

C. Polynomial Function Approximation of Membership Functions

The FOU contains a lot of information of IT2 membership functions carried by an infinite number of embedded type-1 membership functions. In order to take the information of FOU into the stability analysis, in the second method, we first construct a set of embedded type-1 membership functions in polynomial form. Then, the stability analysis is developed based on the embedded type-1 polynomial membership functions which can be effectively dealt within the SOS-based stability analysis, and the stability conditions can be solved numerically using SOSTOOLS [18].

From (14), getting rid of the index l (as no subdomain is required), we have

$$
\begin{aligned}
\dot{V}(t) &= \sum_{i=1}^{p}\sum_{j=1}^{c}\tilde{h}_{ij}(\mathbf{x})\mathbf{z}^T\mathbf{Q}_{ij}(\mathbf{x})\mathbf{z}\\
&= \sum_{i=1}^{p}\sum_{j=1}^{c}(\hat{h}_{ij}(\mathbf{x}) + \hat{\delta}_{ij}(\mathbf{x}))\mathbf{z}^T\mathbf{Q}_{ij}(\mathbf{x})\mathbf{z} \qquad (18)
\end{aligned}
$$

where $\hat{\delta}_{ij}(\mathbf{x}) = \tilde{h}_{ij}(\mathbf{x}) - \hat{h}_{ij}(\mathbf{x})$ denotes the difference between the IT2 membership function $\tilde{h}_{ij}(\mathbf{x})$ and the chosen embedded type-1 membership function $\hat{h}_{ij}(\mathbf{x})$ in a polynomial form. Since $\hat{\delta}_{ij}(\mathbf{x})$ is bounded, there must exist a constant scalar $\overline{\delta}_{ij}$, satisfying $\overline{\delta}_{ij} \geq |\hat{\delta}_{ij}(\mathbf{x})|$ for all $\mathbf{x}$ (or in a domain of interest).

Similar to the stability analysis in the first method, we introduce a polynomial matrix $\mathbf{Y}_{ij}(\mathbf{x}) = \mathbf{Y}_{ij}^T(\mathbf{x}) \geq 0$ requiring $\mathbf{Y}_{ij}(\mathbf{x}) \geq \mathbf{Q}_{ij}(\mathbf{x})$ for all i and j. It can be found that $\overline{\delta}_{ij}\mathbf{Y}_{ij}(\mathbf{x})) \geq \hat{\delta}_{ij}(\mathbf{x})\mathbf{Q}_{ij}(\mathbf{x}))$. It follows from (18) that

$$
\begin{aligned}
\dot{V}(t) &= \sum_{i=1}^{p}\sum_{j=1}^{c}\hat{h}_{ij}(\mathbf{x})\mathbf{z}^T\mathbf{Q}_{ij}(\mathbf{x})\mathbf{z} + \sum_{i=1}^{p}\sum_{j=1}^{c}\hat{\delta}_{ij}(\mathbf{x})\mathbf{z}^T\mathbf{Q}_{ij}(\mathbf{x})\mathbf{z}\\
&\leq \sum_{i=1}^{p}\sum_{j=1}^{c}\mathbf{z}^T(\hat{h}_{ij}(\mathbf{x})\mathbf{Q}_{ij}(\mathbf{x}) + \overline{\delta}_{ij}\mathbf{Y}_{ij}(\mathbf{x}))\mathbf{z}. \qquad (19)
\end{aligned}
$$

Along the same line of derivation, the stability analysis results can be summarized in the following theorem.

Theorem 2: The IT2 PFMB system (9), which is formed by a nonlinear plant represented by the IT2 polynomial fuzzy model (3) and the IT2 polynomial fuzzy controller (7) connected in a closed loop, is guaranteed to be asymptotically stable if there exist polynomial matrices $\mathbf{N}_j(\mathbf{x}) \in \Re^{m\times N}$, $\mathbf{X}(\tilde{\mathbf{x}}) = \mathbf{X}(\tilde{\mathbf{x}})^T \in \Re^{N\times N}$, $\mathbf{Y}_{ij}(\mathbf{x}) = \mathbf{Y}_{ij}(\mathbf{x})^T \in \Re^{N\times N}$, $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, c$, such that the following SOS-based conditions are satisfied:

$$
\begin{aligned}
&\nu^T(\mathbf{X}(\tilde{\mathbf{x}}) - \varepsilon_1(\tilde{\mathbf{x}})\mathbf{I})\nu \text{ is SOS}\\
&\nu^T(\mathbf{Y}_{ij}(\mathbf{x}) - \varepsilon_2(\mathbf{x})\mathbf{I})\nu \text{ is SOS} \quad \forall i, j\\
&\nu^T(\mathbf{Y}_{ij}(\mathbf{x}) - \mathbf{Q}_{ij}(\mathbf{x}) - \varepsilon_3(\mathbf{x})\mathbf{I})\nu \text{ is SOS} \quad \forall i, j\\
&-\nu^T\sum_{i=1}^{p}\sum_{j=1}^{c}\big(\hat{h}_{ij}(\mathbf{x})\mathbf{Q}_{ij}(\mathbf{x}) + \overline{\delta}_{ij}\mathbf{Y}_{ij}(\mathbf{x}) + \varepsilon_4(\mathbf{x})\mathbf{I}\big)\nu \text{ is SOS}
\end{aligned}
$$

where $\nu \in \Re^N$ is an arbitrary vector independent of $\mathbf{x}$; $\hat{h}_{ij}(\mathbf{x})$ is a chosen embedded type-1 membership functions in polynomial form; $\overline{\delta}_{ij}$ is a predefined constant scalar satisfying $\overline{\delta}_{ij} \geq |\hat{\delta}_{ij}(\mathbf{x})|$ for all $\mathbf{x}$ (or in a domain of interest); $\varepsilon_1(\tilde{\mathbf{x}}) > 0$, $\varepsilon_2(\mathbf{x}) > 0$, $\varepsilon_3(\mathbf{x}) > 0$, and $\varepsilon_4(\mathbf{x}) > 0$ are predefined polynomials, $\mathbf{Q}_{ij}(\mathbf{x}) = \tilde{\mathbf{A}}_i\mathbf{X}(\tilde{\mathbf{x}}) + \mathbf{X}(\tilde{\mathbf{x}})\tilde{\mathbf{A}}_i(\mathbf{x})^T + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{N}_j(\mathbf{x}) + \mathbf{N}_j(\mathbf{x})^T\tilde{\mathbf{B}}_i(\mathbf{x})^T - \sum_{k\in\mathbf{K}}\frac{\partial\mathbf{X}(\tilde{\mathbf{x}})}{\partial x_k}\mathbf{A}_i^k(\mathbf{x})\hat{\mathbf{x}}$; and the feedback gains are defined as $\mathbf{G}_j(\mathbf{x}) = \mathbf{N}_j(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}})^{-1}$, $j = 1, 2, \ldots, c$.

Remark 4: Referring to Theorem 2, the number of SOS variables is $(p + 1)c + 1$, and the number of SOS-based stability conditions is $pc + 2$. It can be seen that the number of variables and stability conditions are smaller than those in

Theorem 1, Therefore, the computational burden on solving a feasible solution is reduced. However, when the embedded type-1 membership functions $\hat{h}_{ij}(\mathbf{x})$ are in a higher order polynomial form, the requirement on the numerical accuracy will increase and sometimes makes the computation runs into numerical problems, which hinders the solving of stability conditions.

D. Polynomial Approximation of Subdomains Membership Functions

As discussed above, the advantage of using subdomains of membership functions in the first method is that more information of membership functions can be utilized for the relaxation of stability conditions as the number of subdomains increases, but the drawback is the increase of computational burden. In the second method, embedded type-1 membership functions in polynomial form are utilized, which are in favor of the SOS-based stability analysis, and the polynomial functions contain information of the FOU and IT2 membership functions. However, when only a single embedded type-1 membership function is used for the approximation in the whole operating domain, the order of polynomial functions is, in general, required to be high, resulting in difficulties when using numerical method to obtain a feasible solution to the stability conditions. In order to address these drawbacks, we combine the advantages of both the first and second methods to come up with the third method in this section.

In the third method, we first divide the whole operating domain into some subdomains, as in the first method. Corresponding to each subdomain, embedded type-1 membership functions in polynomial form are employed to extract the information of FOU and IT2 membership functions, as in the second method. Instead of using the embedded type-1 membership functions through the whole domain, the embedded type-1 membership functions are the local ones which can be different from those in other subdomains. Consequently, the local embedded type-1 membership functions will be less complicated compared with the ones in the second method.

It follows from (14) that

$$\begin{aligned}\dot{V}(t) &= \sum_{i=1}^{p}\sum_{j=1}^{c}\mathbf{z}^T\tilde{h}_{ijl}(\mathbf{x})\mathbf{Q}_{ij}(\mathbf{x})\mathbf{z}\\ &= \sum_{i=1}^{p}\sum_{j=1}^{c}\mathbf{z}^T(\hat{h}_{ijl}(\mathbf{x})+\hat{\delta}_{ijl}(\mathbf{x}))\mathbf{Q}_{ij}(\mathbf{x})\mathbf{z}\\ &\le \sum_{i=1}^{p}\sum_{j=1}^{c}\mathbf{z}^T(\hat{h}_{ijl}(\mathbf{x})\mathbf{Q}_{ij}(\mathbf{x})+|\hat{\delta}_{ijl}(\mathbf{x})|\mathbf{Y}_{ijl}(\mathbf{x}))\mathbf{z}\\ &\quad \mathbf{x}\in\Phi_l, \quad l=1,2,\ldots,L \end{aligned} \tag{20}$$

where $\hat{\delta}_{ijl}(\mathbf{x}) = \tilde{h}_{ijl}(\mathbf{x}) - \hat{h}_{ijl}(\mathbf{x})$, and $\hat{h}_{ijl}(\mathbf{x})$ is the local embedded type-1 membership function in polynomial form in the lth subdomain. As $\hat{\delta}_{ijl}(\mathbf{x})$ is bounded, there exists a constant scalar $\overline{\delta}_{ijl}$ satisfying $\overline{\delta}_{ijl} \ge |\hat{\delta}_{ijl}(\mathbf{x})|$ for all $\mathbf{x}$ (or in a domain of interest). Furthermore, with the consideration of the state information in each subdomain, we have

$$\begin{aligned}\dot{V}(t) \le \sum_{i=1}^{p}\sum_{j=1}^{c}\mathbf{z}^T&(\hat{h}_{ijl}(\mathbf{x})\mathbf{Q}_{ij}(\mathbf{x})+\overline{\delta}_{ijl}\mathbf{Y}_{ijl}(\mathbf{x})\\ &+(\mathbf{x}-\underline{\mathbf{x}}_l)^T\mathbf{D}(\overline{\mathbf{x}}_l-\mathbf{x})\mathbf{M}_l(\mathbf{x}))\mathbf{z},\\ &\mathbf{x}\in\Phi_l, \quad l=1,2,\ldots,L. \end{aligned} \tag{21}$$

Along the same line of derivation, the stability analysis results can be summarized in the following theorem.

Theorem 3: The IT2 PFMB system (9), which is formed by a nonlinear plant represented by the IT2 polynomial fuzzy model (3) and the IT2 polynomial fuzzy controller (7) connected in a closed loop, is guaranteed to be asymptotically stable if there exist polynomial matrices $\mathbf{M}_l(\mathbf{x}) = \mathbf{M}_l(\mathbf{x})^T \in \Re^{N\times N}$, $\mathbf{N}_j(\mathbf{x}) \in \Re^{m\times N}$, $\mathbf{X}(\tilde{\mathbf{x}}) = \mathbf{X}(\tilde{\mathbf{x}})^T \in \Re^{N\times N}$, $\mathbf{Y}_{ijl}(\mathbf{x}) = \mathbf{Y}_{ijl}(\mathbf{x})^T \in \Re^{N\times N}$, $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, c$, $l = 1, 2, \ldots, L$, such that the following SOS-based conditions are satisfied:

$$\begin{aligned}&\nu^T(\mathbf{M}_l(\mathbf{x})-\varepsilon_1(\mathbf{x})\mathbf{I})\nu \text{ is SOS} \quad \forall l\\ &\nu^T(\mathbf{X}(\tilde{\mathbf{x}})-\varepsilon_2(\tilde{\mathbf{x}})\mathbf{I})\nu \text{ is SOS}\\ &\nu^T(\mathbf{Y}_{ijl}(\mathbf{x})-\varepsilon_3(\mathbf{x})\mathbf{I})\nu \text{ is SOS} \quad \forall i,j,l\\ &\nu^T(\mathbf{Y}_{ijl}(\mathbf{x})-\mathbf{Q}_{ij}(\mathbf{x})-\varepsilon_4(\mathbf{x})\mathbf{I})\nu \text{ is SOS} \quad \forall i,j,l\\ &-\nu^T\sum_{i=1}^{p}\sum_{j=1}^{c}\Big(\hat{h}_{ijl}(\mathbf{x})\mathbf{Q}_{ij}(\mathbf{x})+\overline{\delta}_{ijl}\mathbf{Y}_{ijl}(\mathbf{x})\\ &\quad+(\mathbf{x}-\underline{\mathbf{x}}_l)^T\mathbf{D}(\overline{\mathbf{x}}_l-\mathbf{x})\mathbf{M}_l(\mathbf{x})+\varepsilon_5(\mathbf{x})\mathbf{I}\Big)\nu \text{ is SOS} \quad \forall l \end{aligned}$$

where $\nu \in \Re^N$ is an arbitrary vector independent of $\mathbf{x}$; $\hat{h}_{ijl}(\mathbf{x})$ is a chosen embedded type-1 membership functions in polynomial form in the lth subdomain; $\overline{\delta}_{ijl}$ is a predefined constant scalar satisfying $\overline{\delta}_{ijl} \ge |\hat{\delta}_{ijl}(\mathbf{x})|$ for all $\mathbf{x}$ (or in a domain of interest); $\varepsilon_1(\mathbf{x}) > 0$, $\varepsilon_2(\tilde{\mathbf{x}}) > 0$, $\varepsilon_3(\mathbf{x}) > 0$, $\varepsilon_4(\mathbf{x}) > 0$, and $\varepsilon_5(\mathbf{x}) > 0$ are predefined scalar polynomials; $\underline{\mathbf{x}}_l$ and $\overline{\mathbf{x}}_l$ are the predefined lower and upper bounds of system state $\mathbf{x}$ in the lth subdomain; $\mathbf{Q}_{ij}(\mathbf{x}) = \tilde{\mathbf{A}}_i\mathbf{X}(\tilde{\mathbf{x}}) + \mathbf{X}(\tilde{\mathbf{x}})\tilde{\mathbf{A}}_i(\mathbf{x})^T + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{N}_j(\mathbf{x}) + \mathbf{N}_j(\mathbf{x})^T\tilde{\mathbf{B}}_i(\mathbf{x})^T - \sum_{k\in\mathbf{K}}\frac{\partial\mathbf{X}(\tilde{\mathbf{x}})}{\partial x_k}\mathbf{A}_i^k(\mathbf{x})\hat{\mathbf{x}}$; and the feedback gains are defined as $\mathbf{G}_j(\mathbf{x}) = \mathbf{N}_j(\mathbf{x})\mathbf{X}(\tilde{\mathbf{x}})^{-1}$, $j = 1, 2, \ldots, c$.

Remark 5: Referring to Theorem 3, the number of SOS variables is $pcL + L + c + 1$, and the number of SOS based stability conditions is $2pcL + 2L + 1$. It can be seen that the number of variables and stability conditions is the same as that in Theorem 1. However, thanks to the introduction of polynomial functions in every subdomain, richer information of membership functions can be included in every subdomain; thus, the number of intervals can be reduced, which means that better performance can be achieved with a smaller value of L. Therefore, the computational burden in Theorem 3 is less than its counterpart in Theorem 1. In addition, the membership functions in subdomains are less complicated than being considered as a whole; it is possible to use low-order polynomial functions to fulfill the approximation task, which avoids the numerical problems that could occur in Theorem 2 when high-order polynomial functions are adopted.

IV. Simulation Examples

Example 1: Let us consider a three-rule polynomial fuzzy model in the form of (9) with $\hat{\mathbf{x}}(\mathbf{x}) = \mathbf{x} = [x_1 \quad x_2]^T$

$$\mathbf{A}_1(x_1) = \begin{bmatrix} 1.59 + 2.45x_1 & -7.29 - 0.89x_1 \\ 0.01 & -0.1 - 0.27x_1^2 \end{bmatrix}$$

$$\mathbf{A}_2(x_1) = \begin{bmatrix} 0.02 - 7.26x_1 - 0.05x_1^2 & -4.64x_1 \\ 0.35 - 0.28x_1 & -0.21 - 1.65x_1^2 \end{bmatrix}$$

$$\mathbf{A}_3(x_1) = \begin{bmatrix} -a + 0.37x_1 - 2.7x_1^2 & -4.33 - 2.73x_1^2 \\ 1.77x_1 & 0.05 - x_1^2 \end{bmatrix}$$

$$\mathbf{B}_1(x_1) = \begin{bmatrix} 1 + 0.37x_1 + 1.28x_1^2 \\ 0 \end{bmatrix}$$

$$\mathbf{B}_2(x_1) = \begin{bmatrix} 8 + 0.23x_1^2 \\ 0 \end{bmatrix}$$

$$\mathbf{B}_3(x_1) = \begin{bmatrix} -b + 6 + 0.72x_1 + 1.55x_1^2 \\ -1 \end{bmatrix}$$

and a and b are constant system parameters. The membership functions are chosen as $\underline{w}_1(x_1) = 1 - 1/(1 + e^{(-x_1+3.5)})$, $\underline{w}_3(x_1) = 1 - 1/(1 + e^{(-x_1-3.5)})$, $\overline{w}_2(x_1) = 1 - \underline{w}_1(x_1) - \underline{w}_3(x_1)$, $\overline{w}_1(x_1) = 1 - 1/(1 + e^{(-x_1+2.5)})$, $\overline{w}_3(x_1) = 1 - 1/(1 + e^{(-x_1-2.5)})$, $\underline{w}_2(x_1) = 1 - \overline{w}_1(x_1) - \overline{w}_3(x_1)$, $\underline{m}_1(x_1) = \max(\min(1, (4.8 - x_1)/10), 0)$, $\overline{m}_1(x_1) = \max(\min(1, (5.2 - x_1)/10), 0)$, $\underline{m}_2(x_1) = 1 - \overline{m}_1(x_1)$, and $\overline{m}_2(x_1) = 1 - \underline{m}_1(x_1)$. The operation max means to pick the largest element and min means to pick the smallest element.

It should be noted that, in this example, the number of fuzzy rules and the membership functions employed for the polynomial fuzzy models and the polynomial fuzzy controllers are different, which can reduce the controller implementation cost when less number of membership functions is employed in the controller.

A. Simulations on Theorem 1

The stability conditions in Theorem 1 are employed to determine the stabilization region of the PFMB control system mentioned above with $60 \le a \le 100$ at the interval of 5 and $20 \le b \le 148$ at the interval of 4.

Referring to Theorem 1, we choose $\varepsilon_1(\mathbf{x}) = \varepsilon_2(\tilde{\mathbf{x}}) = \varepsilon_3(\mathbf{x}) = \varepsilon_4(\mathbf{x}) = \varepsilon_5(\mathbf{x}) = 0.001$; $\mathbf{X}(\tilde{\mathbf{x}})$ as a polynomial of degree 0; $\mathbf{N}_j(x_1)$, $j = 1, 2, \ldots, c$, as a polynomial with monomials in x_1 of degree 0 (for case 1) and degree 2 (for case 2); $\hat{h}_{ijl}$ and δ_{ijl} are defined by the membership functions and the number of subdomains, and they are calculated subdomain by subdomain in the way explained in deduction of Theorem 1; and $\underline{\mathbf{x}}_l$ and $\overline{\mathbf{x}}_l$ are the boundaries of lth subdomain.

The stabilization regions are determined under 5, 10, and 20 subdomains for both cases of polynomial degrees for $\mathbf{N}_j(x_1)$, which are plotted in Figs. 1–3. It is observed that a larger stabilization region can be obtained by higher order polynomial matrices $\mathbf{N}_j(x_1)$ under the same number of subdomains. When more subdomains are employed, larger stabilization regions can be obtained. To verify the results, the phase plots of certain points in the stability regions are shown in Fig. 4. To obtain the phase plots, throughout this example, the membership functions $\tilde{w}_i(x_1)$ and $\tilde{m}_j(x_1)$ used in the simulations are gained from type reduction, where $\underline{\lambda}_1(x_1) = (\sin(5x_1) + 1)/2$, $\overline{\lambda}_1(x_1) = 1 - \underline{\lambda}_1(x_1)$, $\underline{\lambda}_3(x_1) = (\cos(5x_1) + 1)/2$, $\overline{\lambda}_3(x_1) = 1 - \underline{\lambda}_3(x_1)$, and $\underline{\kappa}_j(x_1) = \overline{\kappa}_j(x_1) = 0.5$, $j = 1, 2$. From the property of membership functions in (4) and (8), we have $\tilde{w}_2(x_1) = 1 - \tilde{w}_1(x_1) - \tilde{w}_3(x_1)$. It can be found that all states started from different initial conditions approach $\mathbf{x} = \mathbf{0}$, which means that the system is asymptotically stable. The solutions of the stability conditions in Theorem 1 are found numerically with SOSTOOLS.

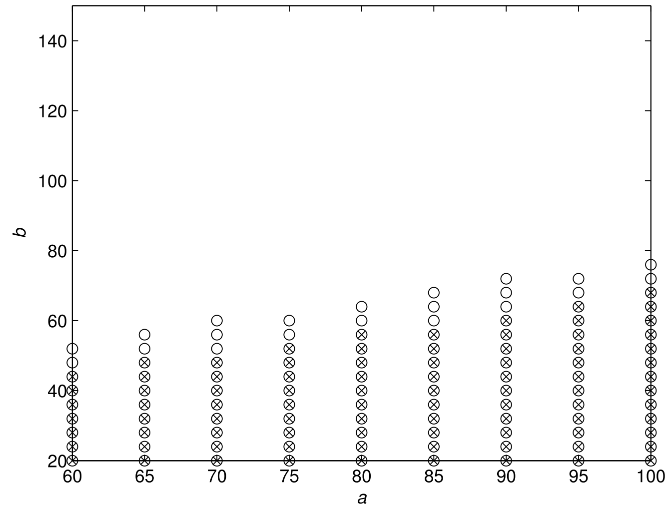

Fig. 1. Stabilization regions given by Theorem 1 with $\mathbf{N}_j(\mathbf{x})$ of degrees 0 and 2 indicated by "×" and "∘," respectively. The number of subdomains is 5.

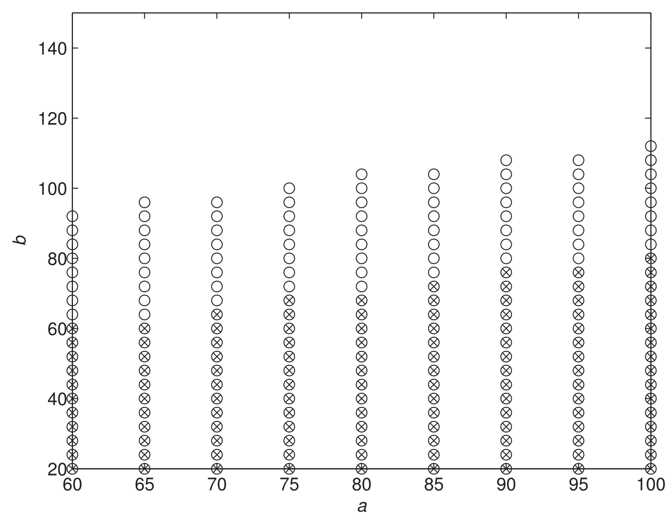

Fig. 2. Stabilization regions given by Theorem 1 with $\mathbf{N}_j(\mathbf{x})$ of degrees 0 and 2 indicated by "×" and "∘," respectively. The number of subdomains is 10.

B. Simulations on Theorem 2

Along the same way in the above simulations, the stability conditions in Theorem 2 are employed to determine the stabilization region of the same PFMB control system. The

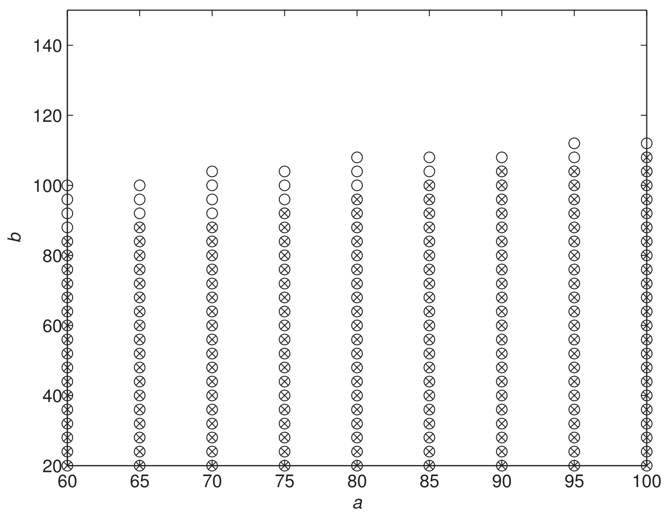

Fig. 3. Stabilization regions given by Theorem 1 with $\mathbf{N}_j(\mathbf{x})$ of degrees 0 and 2 indicated by "×" and "∘," respectively. The number of subdomains is 20.

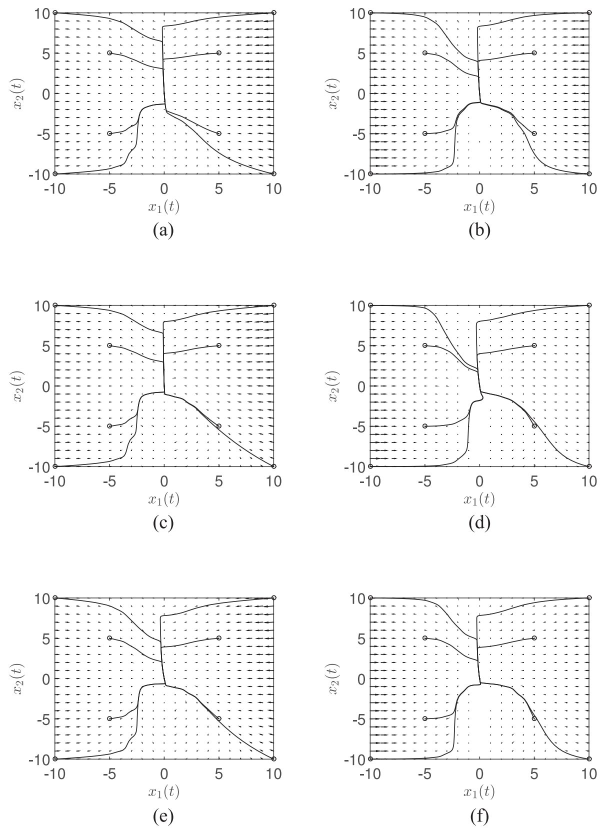

Fig. 4. (a) and (b) Phase plots of $x_1(t)$ and $x_2(t)$ for $a = 80$ and $b = 40$ for Theorem 1 with $\mathbf{N}_j(\mathbf{x})$ of degrees 0 and 2, respectively; the number of subdomains is 5. (c) and (d) Phase plots of $x_1(t)$ and $x_2(t)$ for $a = 60$ and $b = 60$ for Theorem 1 with $\mathbf{N}_j(\mathbf{x})$ of degrees 0 and 2, respectively; the number of subdomains is 10. (e) and (f) Phase plots of $x_1(t)$ and $x_2(t)$ for $a = 80$ and $b = 80$ for Theorem 1 with $\mathbf{N}_j(\mathbf{x})$ of degrees 0 and 2, respectively; the number of subdomains is 20. "∘" indicates the initial condition of $\mathbf{x}$.

simulations have been conducted under both sixth- and eighth-order polynomial approximation functions, while other settings remain the same.

Referring to Theorem 2, we choose $\varepsilon_1(\tilde{\mathbf{x}}) = \varepsilon_2(\mathbf{x}) = \varepsilon_3(\mathbf{x}) = \varepsilon_4(\mathbf{x}) = 0.001$; $\mathbf{X}(\tilde{\mathbf{x}})$ as a polynomial of degree 0; and $\mathbf{N}_j(x_1)$, $j = 1, 2, \ldots, c$, as a polynomial with monomials in x_1 of degree 0 (for case 1) and degree 2 (for case 2); $\hat{h}_{ij}(\mathbf{x})$ and $\overline{\delta}_{ij}$ are defined by the membership functions and the order of the chosen polynomial functions, and they are calculated in the way explained in the deduction of Theorem 2.

The stabilization region is shown in Figs. 5 and 6 for both cases of polynomial degrees for $\mathbf{N}_j(x_1)$. It can be seen that a larger stabilization region can be obtained using higher order polynomial matrices $\mathbf{N}_j(x_1)$. Furthermore, using higher order polynomial functions in the approximation is able to obtain a larger stabilization region. The phase plots under initial conditions are shown in Fig. 7. It can be seen that all states started with different initial positions approach $\mathbf{x} = \mathbf{0}$, which shows the asymptotic stability of the system.

C. Simulations on Theorem 3

The stability conditions in Theorem 3 are employed to determine the stabilization region of the same PFMB control system. Referring to Theorem 3, we choose $\varepsilon_1(\mathbf{x}) = \varepsilon_2(\tilde{\mathbf{x}}) = \varepsilon_3(\mathbf{x}) = \varepsilon_4(\mathbf{x}) = \varepsilon_5(\mathbf{x}) = 0.001$; $\mathbf{X}(\tilde{\mathbf{x}})$ as a polynomial of degree 0; and $\mathbf{N}_j(x_1)$, $j = 1, 2, \ldots, c$, as a polynomial with monomials in x_1 of degree 0 (for case 1) and degree 2 (for case 2); $\hat{h}_{ijl}(\mathbf{x})$ and $\overline{\delta}_{ijl}$ are defined by the membership functions and the order of the chosen polynomial functions as well as the number of subdomains, and they are calculated in the way explained in the deduction of Theorem 3; $\underline{\mathbf{x}}_l$ and $\overline{\mathbf{x}}_l$ are the boundaries of lth subdomain.

With the same settings as above, the simulations have been done under 5, 10, and 20 subdomains using second-order polynomial approximation functions for both cases of polynomial degrees for $\mathbf{N}_j(x_1)$. The stabilization regions are shown in Figs. 8–10. It can be seen that a larger stabilization region can be produced with higher order polynomial matrices $\mathbf{N}_j(x_1)$. The phase plots of system states with different initial conditions are shown in Fig. 11. It can be seen that all states start from different initial conditions approach $\mathbf{x} = \mathbf{0}$. Compared with Theorems 1 and 2, the stability conditions in Theorem 3 are the most relaxed one, which is evident by the largest stability region offered.

The stability analysis of the IT2 PFMB control system is limitedly investigated in the literature. For comparison purposes, we compare the results using the basic stability conditions in Remark 2 that no stability region can be found, which shows the effectiveness of the proposed stability conditions. To probe further, we employ the stability conditions for the type-1 PFMB control system in [13], which offers less conservative stability conditions among some recently published results. The embedded type-1 membership functions taking the average of the lower and upper membership functions are used for the stability conditions in [13]. The stability regions given by the stability conditions in [13] are shown in Fig. 12. It can be seen that the stability regions given by Theorems 1–3 are larger in size, which demonstrates the superiority of the proposed stability analysis results.

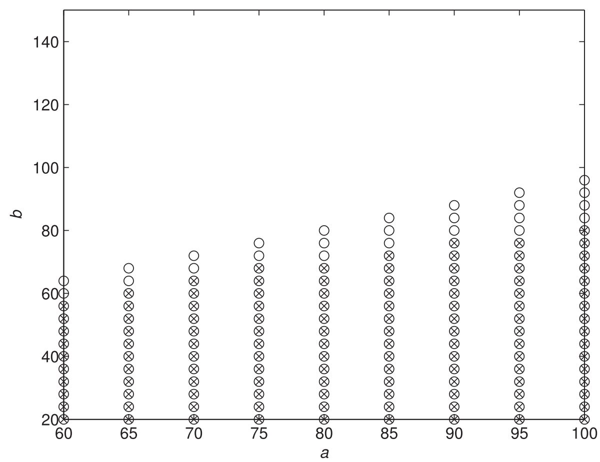

Fig. 5. Stabilization regions given by Theorem 2 with $\mathbf{N}_j(\mathbf{x})$ of degrees 0 and 2 indicated by "×" and "∘," respectively. The order of polynomial functions is 6.

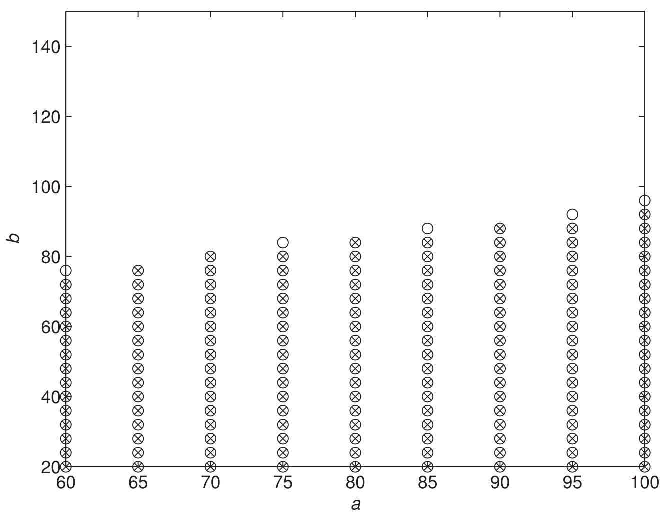

Fig. 6. Stabilization regions given by Theorem 2 with $\mathbf{N}_j(\mathbf{x})$ of degrees 0 and 2 indicated by "×" and "∘," respectively. The order of polynomial functions is 8.

Example 2: In this example, the stability of an inverted pendulum is investigated to verify the applicability of the proposed approaches. The inverted pendulum is an open-loop unstable nonlinear system; therefore, the control task is to apply the developed stability conditions to find the proper feedback gains which can stabilize the inverted pendulum system. The dynamic equation for the inverted pendulum [24] is given by

$$\ddot{\theta} = \frac{g\sin(\theta(t)) - \mathrm{am}_p S\dot{\theta}(t)^2 \sin(2\theta(t))/2 - a\cos(\theta(t))u(t)}{4S/3 - \mathrm{am}_p S\cos^2(\theta(t))} \tag{22}$$

where $\theta(t)$ is the angular displacement of the inverted pendulum, $g = 9.8$ m/s^2, $m_p \in [m_{p_{\min}} \quad m_{p_{\max}}] = [2 \quad 3]$ kg is the mass of the pendulum, $M_c \in [M_{c_{\min}} \quad M_{c_{\max}}] = [8 \quad 16]$ kg is the mass of the cart, $a = \frac{1}{m_p + M_c}$, $2S = 1$ m is the length of the pendulum, and $u(t)$ is the force applied on the cart. m_p and M_c are treated as the parameter uncertainties.

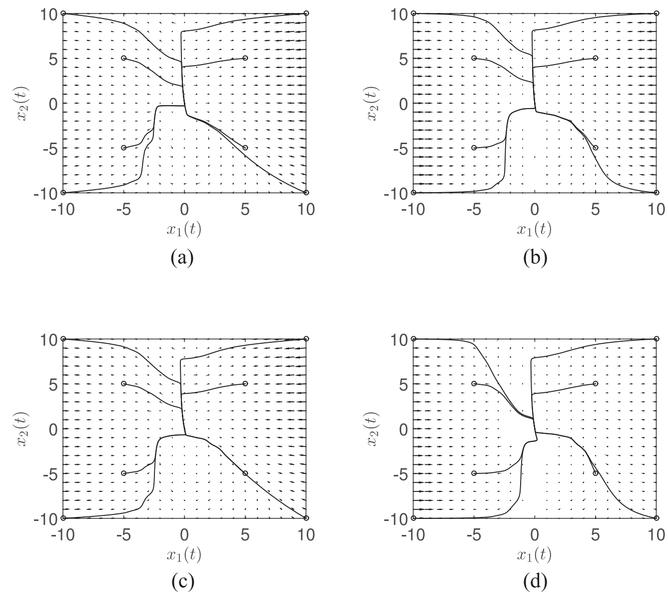

Fig. 7. (a) and (b) Phase plots of $x_1(t)$ and $x_2(t)$ for $a = 70$ and $b = 60$ for Theorem 2 with $\mathbf{N}_j(\mathbf{x})$ of degrees 0 and 2, respectively; the order of the polynomial functions is 6. (c) and (d) Phase plots of $x_1(t)$ and $x_2(t)$ for $a = 70$ and $b = 80$ for Theorem 2 with $\mathbf{N}_j(\mathbf{x})$ of degrees 0 and 2, respectively; the order of the polynomial functions is 8. "∘" indicates the initial condition of $\mathbf{x}$.

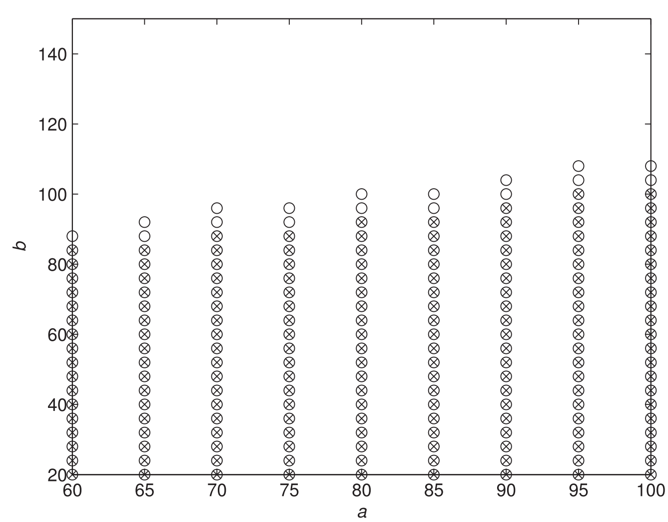

Fig. 8. Stabilization regions given by Theorem 3 with $\mathbf{N}_j(\mathbf{x})$ of degrees 0 and 2 indicated by "×" and "∘," respectively. The number of subdomains and order of polynomial functions are 5 and 2, respectively.

The following four-rule polynomial fuzzy model is adopted to describe the inverted pendulum:

$$\begin{aligned} &\text{Rule } i: \text{ IF } f_1(\mathbf{x}(t)) \text{ is } \tilde{M}_1^i \text{ AND } f_2(\mathbf{x}(t)) \text{ is } \tilde{M}_2^i \\ &\text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t), \\ &\qquad i = 1, 2, 3, 4. \end{aligned} \tag{23}$$

After combining all the fuzzy rules, we have

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{4} \tilde{w}_i \big(\mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t)\big) \tag{24}$$

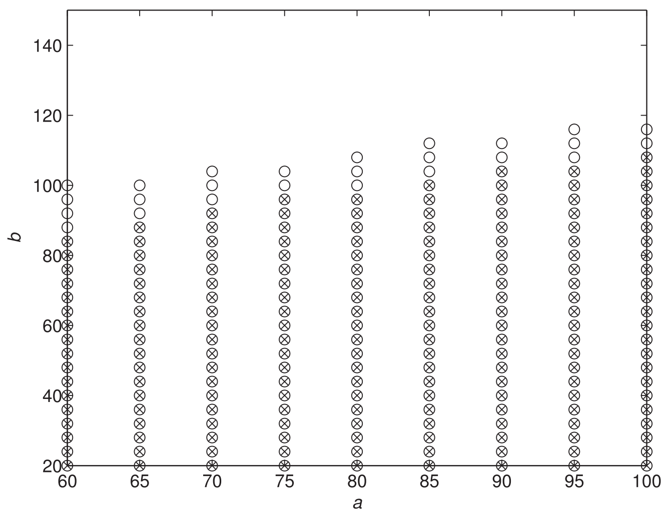

Fig. 9. Stabilization regions given by Theorem 3 with $\mathbf{N}_j(\mathbf{x})$ of degrees 0 and 2 indicated by "×" and "∘," respectively. The number of subdomains and order of polynomial functions are 10 and 2, respectively.

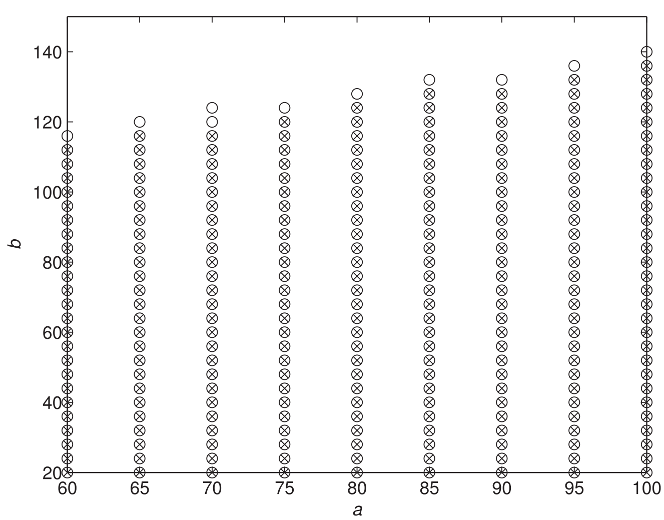

Fig. 10. Stabilization regions given by Theorem 3 with $\mathbf{N}_j(\mathbf{x})$ of degrees 0 and 2 indicated by "×" and "∘," respectively. The number of subdomains and order of polynomial functions are 20 and 2, respectively.

where

$$\hat{\mathbf{x}}(t) = \mathbf{x}(t) = [x_1(t) \quad x_2(t)]^T = [\theta(t) \quad \dot{\theta}(t)]^T$$

$$x_1(t) = \left[\frac{-5\pi}{12} \quad \frac{5\pi}{12}\right], \quad x_2(t) = [-5 \quad 5]$$

$$\mathbf{A}_1 = \mathbf{A}_2 = \begin{bmatrix} 0 & 1 \\ f_{1_{\min}} & 0 \end{bmatrix}, \quad \mathbf{A}_3 = \mathbf{A}_4 = \begin{bmatrix} 0 & 1 \\ f_{1_{\max}} & 0 \end{bmatrix}$$

$$\mathbf{B}_1 = \mathbf{B}_3 = \begin{bmatrix} 0 \\ f_{2_{\min}} \end{bmatrix}, \quad \mathbf{B}_2 = \mathbf{B}_4 = \begin{bmatrix} 0 \\ f_{2_{\max}} \end{bmatrix}.$$

The IT2 membership functions are defined, as shown in Table I.

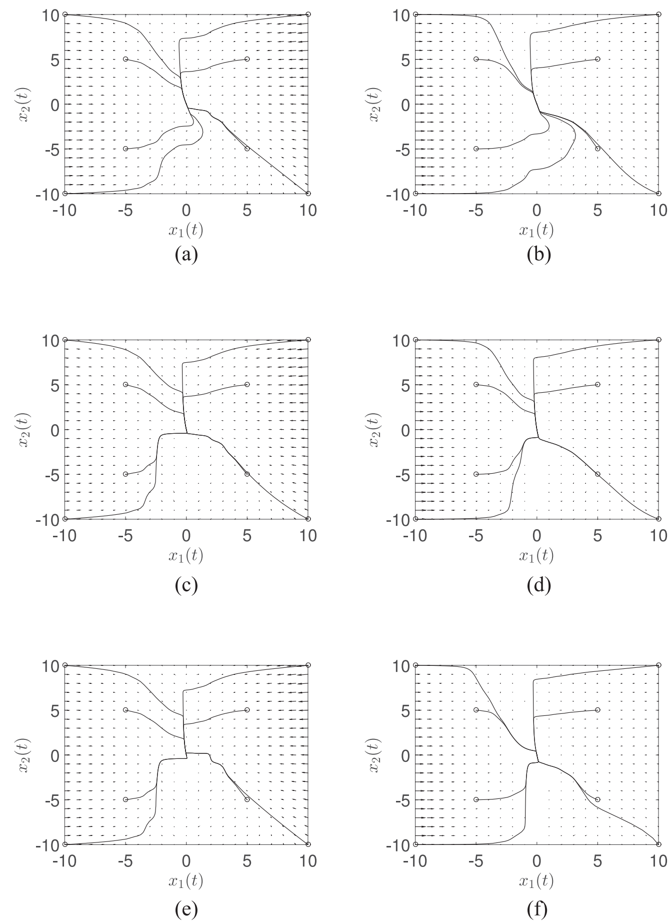

Fig. 11. (a) and (b) Phase plots of $x_1(t)$ and $x_2(t)$ for $a = 60$ and $b = 84$ for Theorem 3 with $\mathbf{N}_j(\mathbf{x})$ of degrees 0 and 2, respectively; the number of subdomains is 5 and the order of polynomial functions is 2. (c) and (d) Phase plots of $x_1(t)$ and $x_2(t)$ for $a = 90$ and $b = 100$ for Theorem 3 with $\mathbf{N}_j(\mathbf{x})$ of degrees 0 and 2, respectively; the number of subdomains is 10 and the order of polynomial functions is 2. (e) and (f) Phase plots of $x_1(t)$ and $x_2(t)$ for $a = 100$ and $b = 136$ for Theorem 3 with $\mathbf{N}_j(\mathbf{x})$ of degrees 0 and 2, respectively; the number of subdomains is 20 and the order of polynomial functions is 2. "∘" indicates the initial condition of $\mathbf{x}$.

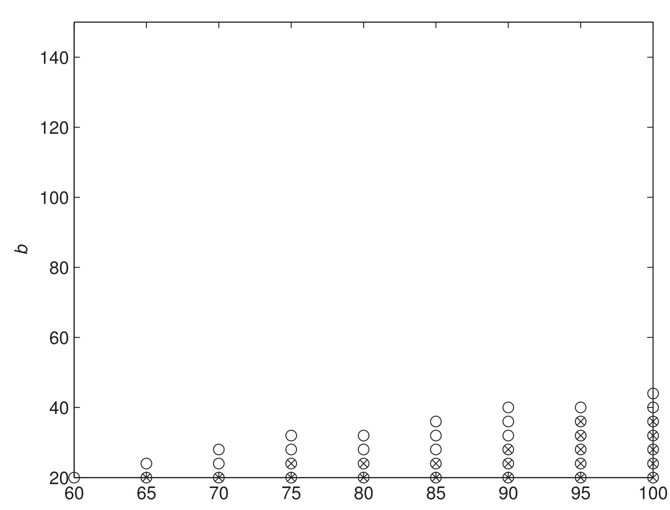

Fig. 12. Stabilization regions given by the method used in [13] with $\mathbf{N}_j(\mathbf{x})$ of degrees 0 and 2 indicated by "×" and "∘," respectively.

TABLE I
LOWER AND UPPER MEMBERSHIP FUNCTIONS FOR THE IT2 FUZZY MODEL OF THE INVERTED PENDULUM

Lower and upper membership functions	
$\underline{\mu}_{\tilde{M}_1^1}(f_1(\mathbf{x}(t))) = \underline{\mu}_{\tilde{M}_1^2}(f_1(\mathbf{x}(t)))$ $= \frac{f_{1\max} - f_1(\mathbf{x}(t))}{f_{1\max} - f_{1\min}}$; $\overline{\mu}_{\tilde{M}_1^3}(f_1(\mathbf{x}(t))) = \overline{\mu}_{\tilde{M}_1^4}(f_1(\mathbf{x}(t)))$ $= \frac{f_1(\mathbf{x}(t)) - f_{1\min}}{f_{1\max} - f_{1\min}}$; with $x_2(t) = 0, m_p = m_{p\max}$ $= 3$ kg and $M_c = M_{c\min} = 8$ kg	$\underline{\mu}_{\tilde{M}_1^1}(f_2(\mathbf{x}(t))) = \underline{\mu}_{\tilde{M}_1^3}(f_2(\mathbf{x}(t)))$ $= \frac{f_{2\max} - f_2(\mathbf{x}(t))}{f_{2\max} - f_{2\min}}$; $\overline{\mu}_{\tilde{M}_1^2}(f_2(\mathbf{x}(t))) = \overline{\mu}_{\tilde{M}_1^4}(f_2(\mathbf{x}(t)))$ $= \frac{f_2(\mathbf{x}(t)) - f_{2\min}}{f_{2\max} - f_{2\min}}$ with $m_p = m_{p\max}$ $= 3$ kg and $M_c = M_{c\max} = 16$ kg
$\overline{\mu}_{\tilde{M}_1^1}(f_1(\mathbf{x}(t))) = \overline{\mu}_{\tilde{M}_1^2}(f_1(\mathbf{x}(t)))$ $= \frac{f_{1\max} - f_1(\mathbf{x}(t))}{f_{1\max} - f_{1\min}}$; $\underline{\mu}_{\tilde{M}_1^3}(f_1(\mathbf{x}(t))) = \underline{\mu}_{\tilde{M}_1^4}(f_2(\mathbf{x}(t)))$ $= \frac{f_2(\mathbf{x}(t)) - f_{2\min}}{f_{2\max} - f_{2\min}}$; with $x_2(t) = x_{2\max}, m_p = m_{p\max}$ $= 3$ kg and $M_c = M_{c\min} = 8$ kg	$\overline{\mu}_{\tilde{M}_1^1}(f_2(\mathbf{x}(t))) = \overline{\mu}_{\tilde{M}_1^3}(f_2(\mathbf{x}(t)))$ $= \frac{f_{2\max} - f_2(\mathbf{x}(t))}{f_{2\max} - f_{2\min}}$ $\underline{\mu}_{\tilde{M}_1^2}(f_2(\mathbf{x}(t))) = \underline{\mu}_{\tilde{M}_1^4}(f_2(\mathbf{x}(t)))$ $= \frac{f_2(\mathbf{x}(t)) - f_{2\min}}{f_{2\max} - f_{2\min}}$; with $m_p = m_{p\min} = 2$ kg and $M_c = M_{c\min} = 8$ kg

In Table I, we have

$$f_1(\mathbf{x}(t)) = \frac{g - \mathrm{am}_p S x_2(t)^2 \cos(x_1(t))}{4S/3 - \mathrm{am}_p S\cos^2(x_1(t))}\left(\frac{\sin(x_1(t))}{x_1(t)}\right)$$

$$f_2(\mathbf{x}(t)) = \frac{-a\cos(x_1(t))}{4S/3 - \mathrm{am}_p S\cos^2(x_1(t))}$$

$$\begin{aligned} f_{1_{\min}} &= -1.8932x_1^2 + 12.0513, f_{1_{\max}} \\ &= -4.3666x_1^2 + 18.4800 \\ f_{2_{\min}} &= -0.0388x_1^4 + 0.1194x_1^2 - 0.1765 \\ f_{2_{\max}} &= -0.0097x_1^4 + 0.0568x_1^2 - 0.0895. \end{aligned}$$

and $f_{1_{\min}}$, $f_{1_{\max}}$, $f_{2_{\min}}$, and $f_{2_{\max}}$ are calculated through a Taylor series-based approach [20]. The lower and upper grades of membership are, respectively, defined as

$$w_i^L(\mathbf{x}(t)) = \underline{\mu}_{\tilde{M}_1^i}(\mathbf{x}(t)) \times \underline{\mu}_{\tilde{M}_2^i}(\mathbf{x}(t))$$

$$w_i^U(\mathbf{x}(t)) = \overline{\mu}_{\tilde{M}_1^i}(\mathbf{x}(t)) \times \overline{\mu}_{\tilde{M}_2^i}(\mathbf{x}(t))$$

for all i.

Based on the IT2 PFMB fuzzy model, a two-rule IT2 polynomial fuzzy controller is adopted to stabilize the inverted pendulum for each control approach we propose in the paper. The following four-rule IT2 polynomial fuzzy controller is adopted to describe the inverted pendulum:

$$\text{Rule } j: \text{ IF } x_1(t) \text{ is } \tilde{N}^j$$
$$\text{THEN } u(t) = \mathbf{G}_j \mathbf{x}(t), \quad j = 1, 2. \tag{25}$$

After combining of all the fuzzy rules, we have

$$u(t) = m_1 \mathbf{G}_1 \mathbf{x}(t) + m_2 \mathbf{G}_2 \mathbf{x}(t). \tag{26}$$

The membership functions are defined: $\overline{m}_1(x_1(t)) = \overline{\mu}_{\tilde{N}^1}(x_1(t)) = \max(\min(\frac{x_1(t)+5\pi/12}{5\pi/12}, \frac{5\pi/12 - x_1(t)}{5\pi/12}), 0)$ $\underline{m}_1(x_1(t)) = \underline{\mu}_{\tilde{N}^1}(x_1(t)) = 0.9 \max(\min(\frac{x_1(t)+5\pi/12}{5\pi/12}, \frac{5\pi/12 - x_1(t)}{5\pi/12}), 0), \overline{m}_2(x_1(t)) = \overline{\mu}_{\tilde{N}^2}(x_1(t)) = 1 - \underline{m}_1(x_1(t))$, $\underline{m}_2(x_1(t)) = \underline{\mu}_{\tilde{N}^2}(x_1(t)) = 1 - \overline{m}_1(x_1(t))$, $m_1(x_1(t)) = \mu_{\tilde{N}^1}(x_1(t))$, $m_2(x_1(t)) = \mu_{\tilde{N}^2}(x_1(t))) = 1 - m_1(x_1(t))$. The type reductions for the controller $\underline{\kappa}_j(x_1(t)) = \overline{\kappa}_j(x_1(t)) = 0.5$, $j = 1, 2$.

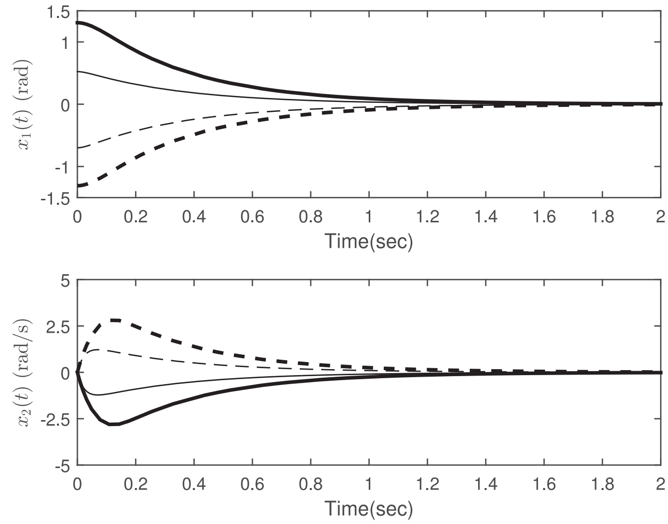

Fig. 13. (Top) Responses of $x_1(t)$. (Bottom) Responses of $x_2(t)$. The number of subdomains is 10.

During the simulations, we set $m_p = 2.5$ kg and $M_c = 12$ kg. Based on Theorem 1, the number of subdomains is 10, and the feedback gains have been achieved as $\mathbf{G}_1 = [751.8281 \quad 249.0395]$, $\mathbf{G}_2 = [2108.07834 \quad 698.2989]$, and $\mathbf{X} = [0.1289 \quad -0.3890; -0.3890 \quad 1.284]$. Based on Theorem 2, the order of the polynomial functions is 4, and the feedback gains have been achieved as $\mathbf{G}_1 = [1677.3300 \quad 530.15796]$, $\mathbf{G}_2 = [1995.9984 \quad 646.2875]$, and $\mathbf{X} = [0.1510 \quad -0.4297; -0.4297 \quad 1.3310]$. Based on Theorem 3, the number of subdomains is 10, and the order of the polynomial functions is 2. The feedback gains have been achieved as $\mathbf{G}_1 = [2012.1242 \quad 613.6520]$, $\mathbf{G}_2 = [8278.3345 \quad 2524.7083]$, and $\mathbf{X} = [0.1218 \quad -0.3995; -0.3995 \quad 1.3430]$. The state response for all the three methods is shown in Figs. 13–15. In those figures, the bold solid curves are under the initial condition $\mathbf{x}(0) = [\frac{5\pi}{12}, 0]$, the regular solid curves are under the condition $\mathbf{x}(0) = [\frac{\pi}{6}, 0]$, the bold dash curves are under the initial condition $\mathbf{x}(0) = [-\frac{5\pi}{12}, 0]$, and the regular dash curves are under the initial condition $\mathbf{x}(0) = [-\frac{\pi}{6}, 0]$. It can be found that all the methods can obtain the proper feedback gains, which can stabilize the inverted pendulum system.

V. CONCLUSION

In this paper, the stability analysis of PFMB control system equipped with IT2 membership functions has been conducted. The imperfectly matched membership functions have been considered and the information of IT2 membership functions has been contained in the analysis, which contributes to further relaxation of the stability conditions. Three approaches for developing the stability conditions of the IT2 PFMB control systems have been proposed. The first approach is able to achieve more relaxed stability conditions through utilizing the information

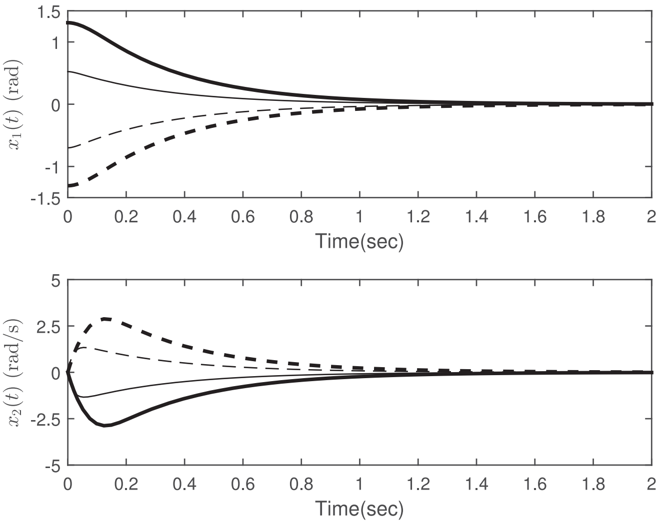

Fig. 14. (Top) Responses of $x_1(t)$. (Bottom) Responses of $x_2(t)$. The order of polynomial functions is 4.

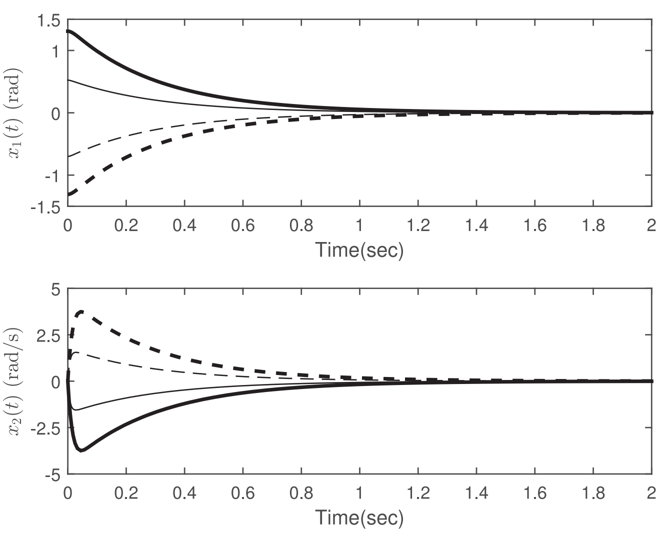

Fig. 15. (Top) Responses of $x_1(t)$. (Bottom) Responses of $x_2(t)$. The number of subdomains is 10, and the order of polynomial functions is 2.

of membership functions in subdomains. The second approach can relax the stability conditions by introducing polynomial approximation functions instead. The third approach can obtain relaxed stability conditions by employing polynomial approximation functions to approximate the original IT2 membership functions in subdomains. Simulation examples have been presented to show the effectiveness of the proposed approaches. The performance of the IT2 PFMB control systems and tracking control based on IT2 PFMB control systems will be considered in the future work.

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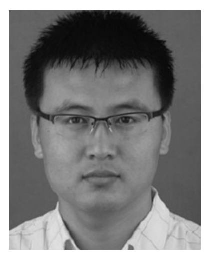

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